Statement calculus

Conclusions will be valid only if they are logically deduced from basic facts by reasoning. Statement calculus is that discipline of knowledge where mathematical logic is detailed.

Logic

Logic provides the rules and techniques for determining whether an argument is valid or not. Logic is used to prove theorems in mathematics and to check the correctness of programs in computer science.

Statements

A declarative sentence that is either true or false but not both is called a *statement*. Statements are also called *propositions*. Generally, in logic, lower case letters $p, q, r \cdots$ are used to denote statements. Examples are

- India is a country
- Delhi is the capital of China
- $1 + 1 = 3$
- John is a boy and Susan is a girl

Here the first and last sentences are true whereas the other two are false. Sentences like Close the door, How lucky you are !, etc are examples that cannot assume True or False values. Hence these are not statements.

Atomic statements

Each individual declaration in a statement is called an *atomic statement.* Atomic statements are also called *primary statements* or *simple statements.* Statements that are not atomic are called *compound* or *composite* or *molecular statements.*

For example, India is a country is an *atomic statement.* 'Ramu is a boy and Lekshmi is a girl' is an example of a compound statement. This contains two atomic statements, viz, Ramu is a boy, Lekshmi is a girl.

Truth tables

If a statement is a true, then we say that the truth value of the statement is True and is denoted by T. If the statement is not true, then the truth value is False and is denoted by F. Tables that present the truth values of statements are called truth tables. The truth table of a compound statement contains the truth values corresponding to all possible truth values of the component atomic statements.

Logical connectives

Statements, whether atomic or not can be connected using connectives. There are specific rules for the truth values of statements connected by the various connectives.

Negation

Let p be a statement. The *negation* of p written $\neg p$ is obtained by negating the statement p. $\neg p$ is read as not p.

For example, the negation statement of 'It is cold today' is 'It is not cold today.' Again, the negation statement of $4+3=7$, is the statement $4+3 \neq 7$. If the truth value of statement is True, then truth value of its negation statement is False and if the truth value is False, then truth value of its negation statement is True.

The truth table for the negation is the following:

p	$\neg p$
T	F
F	T

Conjunction

Let p and q be statements. The *conjunction* of p and q, written $p \wedge q$, is the statement formed by joining the statements p and q using the word 'and'.

Conjunction statement $p \wedge q$ of the statements p and q has truth value T only if both p and q have truth values T. The symbol $\wedge$ is called 'and'.

The truth table of conjunction is

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Notations like, & and $\cdot$ are also used to denote conjunction. It may also be observed that conjunction is a binary operation among statements and also it is commutative in the sense that $p \wedge q$ and $q \wedge p$ have identical truth values.

Consider the statement, p : It is cold today. Another statement q : The season is rainy. The conjunction statement $p \wedge q$ is 'It is cold today and the season is rainy'. Similarly, if p is the statement 'Jack went up the hill' and q is the statement 'Jill went up the hill', then $p \wedge q$: 'Jack and Jill went up the hill' is the conjunction statement.

In the statement 'He opened the book and started reading', 'and' means 'and then'. Hence this is not considered as a conjunction as per the definition.

Disjunction

Let p and q be statements. The *disjunction* of p and q, written $p \vee q$, is the statement formed by joining the statements p and q using the word 'or'.

Disjunction statement $p \vee q$ of the statements p and q has truth value F only if both p and q have truth values F. The symbol $\vee$ is called 'or'.

The truth table of disjunction is

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Disjunction is also a commutative binary operation.

If p denotes the statement 'It is cold today' and q denotes 'The season is rainy', then the disjunction statement $p \vee q$ is 'It is cold today or the season is rainy'. Similarly, if p is the statement 'Jack went up the hill' and q is the statement 'Jill went up the hill', their disjunction statement $p \vee q$ is 'Jack or Jill went up the hill'.

Example

Using the statements r: Mark is rich, h: Mark is happy, write the following statements in the symbolic form:

(a) Mark is poor but happy.

(b) Mark is rich or unhappy.

(c) Mark is neither rich nor happy.

(d) Mark is poor or he is both rich and unhappy.

Ans:

(a) $\neg r \wedge h$.

(b) $r \vee \neg h$.

(c) $\neg r \wedge \neg h$.

(d) $\neg r \vee (r \wedge \neg h)$.

Example

Let p and q be two statements. Construct the truth tables for $(p \vee q) \vee \neg p$
Ans: The following is the required truth table.

p	q	$p \vee q$	$\neg p$	$(p \vee q) \vee \neg p$
T	T	T	F	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	T

Exercise

Construct the truth tables for the following statements.

(a) $p \vee \neg q$

(b) $p \wedge \neg p$

(c) $\neg(\neg p \vee \neg q)$

(d) $p \wedge (p \vee q)$

(e) $p \vee (q \wedge r)$.

Conditional statements

If p and q are two statements, then the statement $p \to q$, which is read as 'if p then q ' is called a conditional statement. $p \to q$ has a truth value F only when p is true and q is false. Thus the truth table for $p \to q$ is the following:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

In the statement $p \to q$, p is called the *antecedent* and q is called the *consequent*. Note that there need not be any relation between p and q in order to form the conditional statement $p \to q$. The statement $p \to q$ can also be expressed like

(1) q is necessary for p

(2) p is sufficient for q

(3) q if p

(4) p only if q

(5) p implies q.

Example

Write the statement 'If either Jerry takes Calculus or Ken takes Sociology, then Larry will take English' in symbolic form.
Ans: Let p: Jerry takes Calculus, q: Ken takes Sociology, r: Larry takes English. Then the statement 'Either Jerry takes Calculus or Ken takes Sociology' is denoted like $p \vee q$. The complete statement takes the form $(p \vee q) \to r$.

Example

'If Tom and Jerry went up the hill, then they would hunt the treasure'. Write this statement in symbolic form.

Ans: Let p denotes the statement'Tom went up the hill', q denotes 'Jerry went up the hill' and r denotes 'Tom hunts the treasure' and s denotes 'Jerry hunts the treasure'. Then in symbols, the combined statement 'If Tom and Jerry went up the hill, then they would hunt the treasure', can be written in the form $(p \wedge q) \rightarrow (r \wedge s)$.

Example

Write in symbols the statement, 'The crop will be destroyed if there is a flood'.

Ans: If p denotes the statement 'The crop will be destroyed' and q denotes the statement 'There is a flood', then the combined statement can be written in the form $q \rightarrow p$.

Example

Construct the truth table for the combination statement $(p \rightarrow q) \wedge (q \rightarrow p)$.

Ans: Truth table for $(p \rightarrow q) \wedge (q \rightarrow p)$ is the following:

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Bi conditional statement

If p and q are two statements, then the statement $p \rightleftharpoons q$ which is read as p if and only if q is called bi conditional statement. This statement is true only when both p and q have identical truth values. The truth table in this case is as follows:

p	q	$p \rightleftharpoons q$
T	T	T
T	F	F
F	T	F
F	F	T

Notice that the truth tables for $p \rightleftharpoons q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$ are identical. This means that $p \rightleftharpoons q$ is a conjunction of the conditional statements $p \rightarrow q$ and $q \rightarrow p$.

Example

Write in symbols the statement, 'The number x^2 is positive if and only x is a real number.'
Ans: Let p denotes the statement 'The number x^2 is positive' and q denotes the statement 'x is a real number'. Here the given statement means 'if the number x^2 is positive, then x is a real number' and 'if x is a real number, then x^2 is positive'. In symbols '$(p \to q) \wedge (q \to p)$.' This is expressed as $p \rightleftharpoons q$.

Statement variables and formula

Variables used to denote statements are called *statement variables.* Letters $p, q, r, \cdots$, are used to denote statement variables. An expression involving statements and logical connectives constitute *statement formula.*

Example

Construct the truth table for the statement formula $A : \neg(p \wedge q) \rightleftharpoons (\neg p \wedge \neg q)$.
Ans: The truth table in this case is

p	q	$p \wedge q$	$\neg(p \wedge q)$	$(\neg p \wedge \neg q)$	A
T	T	T	F	F	T
T	F	F	T	F	F
F	T	F	T	F	F
F	F	T	F	T	F

Exercises

1 Construct the truth tables for the following formulas:

(a) $((p \to q) \wedge (q \to r)) \to (p \to r)$.

(b) $(p \to q) \rightleftharpoons (\neg p \to \neg q)$

(c) $(p \rightleftharpoons q) \rightleftharpoons ((p \wedge q) \vee (\neg p \wedge \neg q))$

2 Given the truth values of p and q as T and those of r and s as F, find the truth values of $(p \vee (q \to (r \wedge \neg p))) \rightleftharpoons (q \vee \neg s)$.

Well-formed formulas

Well formed formula (wff) is defined recursively as follows:

(a) A statement variable standing alone is a well formed formula

(b) If A and B are well formed formulas, then $\neg A$, $(A \wedge B)$,$(A \vee B)$,$(A \to B)$ and $(A \rightleftharpoons B)$ are all well formed formulas,

(c) A string of symbols containing statement variable, connectives and parenthesis is a well formed formula if and only if it can be obtained by finitely many applications of the rules (a) and (b).

Examples

Statements like $\neg(p \vee q), \neg(p \wedge q), (p \rightarrow (p \vee q)), (((p \rightarrow q) \wedge (q \rightarrow r) \rightleftharpoons (p \rightarrow r))$ are all examples wff.

$\neg p \wedge q$ is not a wff. This can be a wff only if it can be put in the form $\neg(p \wedge q)$ or $(\neg p \wedge q)$. Similarly, $(p \rightarrow q) \rightarrow (\neg q)$ is not a wff. This becomes wff if it is of the form $((p \rightarrow q) \rightarrow (\neg q))$.

Tautology

A statement formula which is true regardless of the truth values assigned to the variables, is called a *tautology*.

If p is any statement variable, then $(p \vee \neg p)$ is a tautology. Generally, T denotes a tautology.

Contradiction

A statement formula which is false regardless of the truth values assigned to the variables, is called a *contradiction*.

If p is any statement variable, then $(p \wedge \neg p)$ is a contradiction. Generally,F denotes a contradiction.

Exercises

Check whether the following are well formed formulas, contradictions, tautologies etc.

(1) $(p \rightarrow (p \vee q))$

(2) $((p \rightarrow (\neg p)) \rightarrow \neg p)$

(3) $((p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r)))$.

Implication and equivalence of formulas

A statement formula A is said to *logically imply* a statement formula B if $A \rightarrow B$ is a tautology. If A logically implies B, then symbolically we write $A \implies B$

Two formulas A and B are said to be *logically equivalent* if $A \rightleftharpoons B$ is a tautology. If A is logically equivalent to B, then symbolically we write $A \iff B$.

From the following table, it follows that if p is any statement, then $\neg\neg p$ and p are equivalent.

p	$\neg p$	$\neg\neg p$	$p \rightleftharpoons \neg\neg p$
T	F	T	T
F	T	F	T

Similarly it can be shown that

1. $p \vee p$ is equivalent to p
2. $(p \wedge p) \vee q$ is equivalent to q.

Theorem 1. *Let p and q be two statements. Then $(p \to q)$ is logically equivalent to $(\neg p \vee q)$.*

Proof. *Let A denotes the statement formula $(p \to q) \rightleftharpoons (\neg p \vee q)$. Then proof of the theorem follows from the following table:*

p	q	$\neg p$	$(p \to q)$	$(\neg p \vee p)$	A
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Equivalent formulas

By constructing appropriate truth tables, the following equivalences may be established:

(1) $p \vee p \iff p$, $p \wedge p \iff p$. (Idempotent laws)

(2) $(p \vee q) \vee r \iff p \vee (q \vee r)$, $(p \wedge q) \wedge r \iff p \wedge (q \wedge r)$. (Associative)

(3) $(p \vee q) \iff q \vee p, (p \wedge q) \iff q \wedge p$. (Commutative)

(4) $p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r), p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$. (Distributive)

(5) $p \vee (p \wedge q) \iff p$, $p \wedge (p \vee q) \iff p$. (Absorption law)

(6) $\neg(p \vee q) \iff (\neg p \wedge \neg q)$, $\neg(p \wedge q) \iff (\neg p \vee \neg q)$. (De Morgan's laws)

(7) $p \vee F \iff p, p \wedge T \iff p$.

(8) $p \vee T \iff T, p \wedge F \iff F$ where T and F denote tautologies and contradictions respectively.

Example

In front of a restaurant it is written 'Good food is not cheap'. and in front of another restaurant it is written 'Cheap food is not good'. Do they mean the same of different ?

Ans: Let p: food is good and q: food is cheap. If the given statements are denoted by A and B, then $A: \ p \to \neg q$ and $B: \ q \to \neg p$. The truth table here is as follows:

p	q	$\neg p$	$\neg q$	A	B	$A \rightleftharpoons B$
T	T	F	F	F	F	T
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

It follows that $A \rightleftharpoons B$ is a tautology. Hence the two statements mean the same.

Example

Prove that $p \to (q \vee r) \iff (p \to q) \vee (p \to r)$.
Ans: We have

$$\begin{aligned} p \to (q \vee r) &\iff \neg p \vee (q \vee r) \\ \iff (\neg p \vee \neg p) \vee (q \vee r) &\iff \neg p \vee (\neg p \vee q) \vee r \\ \iff \neg p \vee (q \vee \neg p) \vee r &\iff (\neg p \vee q) \vee (\neg p \vee r) \\ &\iff (p \to q) \vee (p \to r). \end{aligned}$$

Hence the result.

Example

Show that $(\neg p \wedge (\neg q \wedge r)) \vee (q \wedge r) \vee (p \wedge r) \iff r$.
Ans: We have

$$\begin{aligned} (\neg p \wedge (\neg q \wedge r)) \vee (q \wedge r) \vee (p \wedge r) &\iff (\neg p \wedge (\neg q \wedge r)) \vee ((q \vee p) \wedge r)) \\ \iff ((\neg p \wedge \neg q) \wedge r) \vee ((q \vee p) \wedge r)) &\iff (\neg (p \vee q) \vee (p \vee q)) \wedge r \\ \iff T \wedge r \quad \text{(where } T \text{ is a tautology)} &\iff r. \end{aligned}$$

Hence the result.

Example

Show that $((p \vee q) \wedge \neg(\neg p \wedge (\neg q \vee \neg r))) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$ is a tautology.
Ans: We have

$$\begin{aligned} \neg(\neg p \wedge (\neg q \vee \neg r)) &\iff p \vee \neg(\neg q \vee \neg r) \\ &\iff p \vee \neg(\neg(q \wedge r)) \\ &\iff p \vee (q \wedge r) \\ &\iff (p \vee q) \wedge (p \vee r) \\ \text{Hence} \quad ((p \vee q) \wedge \neg(\neg p \wedge (\neg q \vee \neg r))) &\iff (p \vee q) \wedge (p \vee q) \wedge (p \vee r) \\ &\iff (p \vee q) \wedge (p \vee r) \\ \text{Using De Morgan's laws} \qquad \neg p \wedge \neg q &\iff \neg(p \vee q) \quad \text{and} \\ \neg p \wedge \neg r &\iff \neg(p \vee r) \\ \therefore \quad (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r) &\iff \neg(p \vee q) \vee \neg(p \vee r) \\ &\iff \neg((p \vee q) \wedge (p \vee r)) \end{aligned}$$

Therefore given expression is of the form $(A \vee \neg A)$ where $A : (p \vee q) \wedge (p \vee r)$. This is a tautology.

Exercise

1 Without using truth tables, prove that the following are tautologies:

(a) $\neg(\neg p \wedge q) \vee q$.

(b) $(\neg p \wedge q) \rightarrow (\neg(q \rightarrow p))$.

(c) $(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$.

(d) $p \rightarrow (q \vee r) \iff (p \rightarrow q) \vee (p \rightarrow r)$.

2 Prove that $p \rightarrow (q \rightarrow r) \iff p \rightarrow (\neg q \vee r) \iff (p \wedge q) \rightarrow r$.

3 Prove that $\neg(p \rightleftharpoons q) \iff (p \vee q) \wedge \neg(p \wedge q)$.

4 Obtain equivalent formula using connectives $\wedge$ and $\neg$, $(p \rightleftharpoons (q \rightarrow (r \vee p)))$.

Converse, inverse and contra-positive statements

Let p and q be two statements. Then

(a) The statement $q \rightarrow p$ is the *converse* of $p \rightarrow q$.

(b) The statement $\neg p \rightarrow \neg q$ is the *inverse* of $p \rightarrow q$.

(c) The statement $\neg q \rightarrow \neg p$ is the *contra-positive* of $p \rightarrow q$.

Example

For the statement 'If the flood destroys my house or fire destroys my house, then my insurance company will pay me', write the converse, inverse and contra-positive statements.

Ans: Let the statement 'Flood destroys my house', 'Fire destroys my house' and 'Insurance company pays me' be denoted by p, q and r respectively. Then given statement is $(p \vee q) \rightarrow r$.

Converse of this statement is $r \rightarrow (p \vee q)$. This means ' If the insurance company pays me, then either flood destroys my house or fire destroys my house.'

Inverse of this statement is $\neg(p \vee q) \rightarrow \neg r$. This statement is equivalent to $(\neg p \wedge \neg q) \rightarrow \neg r$.

This means ' If flood and fire do not destroy my house then the insurance company does not pay me.

Contra-positive of the statement is $\neg r \rightarrow \neg(p \vee q)$, which is equivalent to $\neg r \rightarrow (\neg p \wedge \neg q)$.

This means ' If the insurance company does not pay me, then flood does not destroy my house and fire does not destroy my house.'

Exercises

1. For the statement 'If there is a will, there is a way' write the converse, inverse and contra-positive statements.

2. For the statement 'If there is rainbow, then there is cloud in the sky', write the converse, inverse and contra-positive statements.

Duals of logical formulas

Two logical formulas A and A^* are duals of each other if either one can be obtained from the other by replacing $\vee$ by $\wedge$ and $\wedge$ by $\vee$. The connectives $\vee$ and $\wedge$ are considered to be duals of each other.

If A contains the special variables T or F, then A^* is obtained by replacing T by F and F by T.

Examples

Write the duals of the statements

(1) $(p \vee q) \wedge r$

(2) $(p \wedge q) \vee r$

(3) $(p \vee \neg(q \vee \neg s))$

(4) $\neg(p \vee q) \vee (p \wedge \neg(q \vee \neg r))$.

Ans:

(1) $(p \wedge q) \vee r$

(2) $(p \vee q) \wedge r$

(3) $(p \wedge \neg(q \wedge \neg s))$

(4) $\neg(p \wedge q) \wedge (p \vee \neg(q \wedge \neg r))$.

Validity of arguments

A finite sequence $H_1, H_2, \cdots, H_{n-1}, C$ of statement formulas is called an *argument*. Here $H_1, H_2, \cdots, H_{n-1}$ are called *premises* and C is called *conclusion*.

An argument $H_1, H_2, \cdots, H_{n-1}, C$ is called *logically valid* if the statement formula $H_1 \wedge H_2 \wedge \cdots \wedge H_{n-1} \to C$ is a tautology.

Example

If $H_1 : \neg p \vee r, H_2 : \neg(q \wedge \neg r), H_3 : r, C : \neg p$, prove that the conclusion C follows from the premises H_1, H_2 and H_3.

Ans: Here let H denotes $H_1 \wedge H_2 \wedge H_3$, then the truth table is the following:

p	q	r	$\neg p$	$\neg r$	H_1	$q \wedge \neg r$	H_2	$H_1 \wedge H_2$	H	$H \to C$
T	T	T	F	F	T	F	T	T	F	T
T	T	F	F	T	T	T	F	F	F	T
T	F	T	F	F	F	F	T	F	F	T
T	F	F	F	T	F	F	T	F	F	T
F	T	T	T	F	T	F	T	T	F	T
F	T	F	T	T	T	T	F	F	F	T
F	F	T	T	F	T	F	T	T	F	T
F	F	F	T	T	T	T	T	T	T	T

It follows that the statement formula is a tautology and hence the argument is logically valid.

Example

Check the validity of the argument 'If Sheila solved seven problems correctly, then Sheila obtained the grade A. Sheila solved seven problems correctly. Therefore Sheila obtained the grade A.

Ans: Let p and q denote the statements ' Sheila solved seven problems correctly' and 'Sheila obtained the grade A' respectively. Hence the given argument is $(p \to q), p, q$.

Let $A: \ ((p \to q) \wedge p) \to q$. Then the truth table is as follows:

p	q	$p \to q$	$(p \to q) \wedge p$	A
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

This is a valid argument since $((p \to q) \wedge p) \to q$ is a tautology.

Example

Check the validity of the argument 'If Peter solved seven problems correctly, then Peter obtained the grade A. Peter obtained the grade A. Therefore Peter solved seven problems correctly.

Ans: Let the notations be p: Peter solved seven problems correctly, q: Peter obtained the grade A. Hence the given argument is $(p \to q), q, p$.

Let $B: \ ((p \to q) \wedge q) \to p$. Then the truth table is as follows:

p	q	$p \to q$	$(p \to q) \wedge q$	B
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

This is not a valid argument since $((p \to q) \wedge q) \to p$ is not a tautology.

Example

Test the validity of the arguments: I can graduate only if I have a G.P.A of 3.5. Either I am smart or I do not have a G.P.A of 3.5. I did not graduate. Hence I am not smart.

Ans: Let the statements be denoted as as p: I do graduate, q: I have a G.P.A of 3.5, r: I am smart. The given statements in symbols are $q \to p, r \vee \neg q, \neg p, \neg r$. We have to check whether $(q \to p) \wedge (r \vee \neg q) \wedge \neg p \to \neg r$ is a tautology. Now $(q \to p) \wedge (r \vee \neg q) \wedge \neg p \iff (\neg q \vee p) \wedge (r \vee \neg q) \wedge \neg p \iff ((p \vee \neg q) \wedge (r \vee \neg q) \wedge \neg p \iff ((p \wedge r) \vee \neg q) \wedge \neg p$. Hence it is sufficient to check whether $(((p \wedge r) \vee \neg q) \wedge \neg p) \to \neg r$

is a tautology. Let $A : (p \wedge r), B : (p \wedge r) \vee \neg q, C : (((p \wedge r)\neg q) \wedge \neg p)$ and $D : (((p \wedge r)\neg q) \wedge \neg p) \rightarrow \neg r$, then the truth table is as follows:

p	q	r	$\neg p$	$\neg q$	$\neg r$	A	B	C	D
T	T	T	F	F	F	T	T	T	F
T	T	F	F	F	T	F	F	F	T

Since the entries in the last columns of the first two rows are different this is not a tautology. Hence the arguments are not valid.

Exercises

1 Show that the conclusion C follows from the premises $H_1, H_2, \cdots, H_{n-1}$ in the following cases:

(a) $H_1 : p \rightarrow q, C : p \rightarrow (p \wedge q)$.

(b) $H_1 : p \rightarrow q, H_2 : q \rightarrow r, C : p \rightarrow r$.

2 Determine whether the conclusion C follows from the premises $H_1, H_2, \cdots, H_{n-1}$ in the following cases:

(a) $H_1 : p \rightarrow q,\ H_2 : \neg q,\ C : p$.

(b) $H_1 : p \vee q,\ H_2 : p \rightarrow r,\ H_3 : q \rightarrow r,\ C : r$.

3 Check whether the following arguments are logically valid.

(a) If it rains, the prices of vegetables go up. The prices of vegetables go up. So it rains.

(b) If I save money, I will buy a house. I did not buy a house. Therefore, I did not save money.

(c) Anne plays golf or Anne plays basketball. Therefore Anne plays golf.

(d) If Rita works hard and has talent, then she will get a good job. If she gets a good job, then she will be happy. Hence if Rita is not happy, then she did not work hard or she does not have talent.

(e) For a particular real number x : x is positive or negative. If x is positive, then $x^2 > 0$. If x is negative, then $x^2 > 0$. Therefore $x^2 > 0$.

(f) If Sharu misses many shootings, then his film will be a flop. If Sharu's film flops, then he fails to act. If he signs for many films, then he does not fail to act. Sharu misses many shootings and signs for many films.

Predicate Calculus

In the statement calculus that we have discussed so far, our concerns were atomic statements, logical connectives and their basic properties. But these connectives do not consider the situation when the atomic statements have some features in common. In order to investigate these questions, the concept of predicates in atomic statements is introduced. The logic based on predicates in statements is called predicate logic.

Predicates

Consider the statements

- John is a bachelor.
- Smith is a bachelor.

Here these two statements have a common property that both of them are bachelors. The part 'is a bachelor' in these statements is called a *predicate*.

Symbolic representation

In predicate calculus, the predicates are denoted by capital letters and the subjects are denoted by small letters.

In the statements 'John is a bachelor' and 'Smith is a bachelor' we have noticed that 'is a bachelor' is the predicate. If this predicate is denoted by the letter B and the subjects John and Smith respectively by j and s, then $B(j)$ means j satisfy B and $B(s)$ means s satisfy B. Thus the combined statement here is $B(j) \wedge B(s)$.

$m-$place predicate

A predicate requiring one name is called *one place predicate*. Similarly that requiring two names is called a *two place predicate*. In general a predicate requiring m names is called *$m-$place predicate*.

The predicate B which means ' is a bachelor' is a one place predicate. If L denotes the predicate ' is less than', then L is a two place predicate. Again, if S denotes the predicate ' sits between', then S is an example of 3 place predicate.

Simple and Compound statement functions

If the subject in a predicate expression is a variable, then such an expression is called a *statement function*.

Generally, if $B(x)$ means 'x is a bachelor', here the subject is a variable. $B(x)$ is an example of a statement function. $L(x, y)$ which means 'x is less than y' is another example of a statement function. Notice that a statement function is not a statement.

An expression consisting of a predicate symbol and an individual variable is called a *simple statement function* of one variable.

In the above discussions, $B(x)$ is a simple statement function of one variable. $L(x, y)$ is a simple statement function of two variables.

When one or more simple statement functions are combined by logical connectives, the resulting expression is called *compound statement function*.

For example $B(x) \wedge L(x, y)$ is an example of compound statement function. Again, $\neg B(x)$ is another example of a compound statement function.

Substitution instance

A statement resulting from statement function by replacing the variables by the names of objects is called a*substitution instance* of the statement function. Note that the substitution instance of the statement function is a statement.

For example, in the statement function $B(x)$, when x is replaced by j, then the result $B(j)$ is a substitution instance of $B(x)$.

Similarly $L(2,3)$ which means 2 is less than 3 is a substitution instance of $L(x,y)$.

Universal quantifier

The expression 'for all' that occur in statements is called *universal quantifier* and is denoted by $\forall$ or ( ). Equivalent to 'for all' we can also have expressions 'for every' and 'for any'. For example consider the two statements

- Flowers are beautiful
- Every integer is either positive or negative.

These statements can be expressed in the form

- For all x, if x is a flower, then x is beautiful
- For all x, if x is an integer, then x is either positive or negative.

Here if $F(x)$ denotes x is a flower, $B(x)$ denotes that x is beautiful, then the first statement may be written like 'For all x if $x \in F(x)$, then $x \in B(x)$. In symbols,

$$(x)(F(x) \rightarrow B(x)) \text{ or } (\forall x)(F(x) \rightarrow B(x))$$

In the second condition, if $I(x)$ denotes x is an integer, $P(x)$ denotes that x is positive or negative, then the statement can me written like 'For all x if $x \in I(x)$, then $x \in P(x)$. In symbols,

$$(x)(I(x) \rightarrow P(x)) \text{ or } (\forall x)(I(x) \rightarrow P(x))$$

In statement functions there can be more than one universal quantifier. For example let $G(x,y)$ stands for the statement function 'x is taller than y.' Then we write 'for all x and y if x is taller than y, then y is not taller than x.' In notations

$$(\forall x)(\forall y)(G(x,y) \rightarrow \neg G(y,x))$$

Existential quantifier

The expression 'for some' that occur in statements is called *existential quantifier* and is denoted by $\exists$. Equivalent to 'for some' we can also have expressions 'there is at least one' and 'there exists some'. For example consider the two statements

- There exists a man.

- Some real numbers are rational.

These statements can be expressed in the form

- For some x, x is a man

- For some x, x is a real number and x is a rational number

Here if $M(x)$ denotes x is a man, then the first statement may be written like 'For some x, $x \in M(x)$. In symbols,

$$(\exists x)(M(x)).$$

In the second condition, if $R(x)$ denotes that x is a real number, $Q(x)$ denotes that x is a rational number, then this statement can be written like 'For some x $x \in R(x)$ and $x \in Q(x)$. In symbols,

$$(\exists x)(R(x) \wedge Q(x))$$

Negation of Quantified statements

Negation of the statement '$(\forall x)P(x)$' is '$(\exists x)\neg P(x)$.' In notations

$$\neg(\forall x)P(x) \iff (\exists x)\neg P(x).$$

Similarly

$$\neg(\exists x)P(x) \iff (\forall x)\neg P(x).$$

Well formed formula of predicate calculus

A statement formula that cannot be further splitted to statement formulas is called *atomic statement formula*. For example, if $M(x)$ denotes the statement formulas that x is a man, then this is an atomic statement formula. Well formed formula (wff) in predicate calculus is defined recursively as follows:

(1) An atomic statement formula is a well formed formula.

(2) If A and B are well formed formulas, then $\neg A$, $(A \wedge B)$,$(A \vee B)$,$(A \rightarrow B)$ and $(A \rightleftharpoons B)$ are all well formed formulas,

(3) If A is a well formed formula and x is a variable, then $(\forall x)A$ and $(\exists x)A$ are well formed formulas.

(4) Only those formulas obtained by using (1) to (3) are well formed formulas.

Free and bound variables

In any formula, a part of the form $(\forall x)P(x)$ or $(\exists x)P(x)$ is called an $x-$*bound* part of the formula. Any occurrence of x in an $x-$bound part of the formula is called a *bound occurrence* of x. Any occurrence of x or any variable that is not a bound occurrence is called *free occurrence.*

The formula $P(x)$ in $(\forall x)P(x)$ or in $(\exists x)P(x)$ is called the *scope* of the quantifier.

For example, in the formula $(\forall x)P(x, y)$, we notice that $P(x, y)$ is the scope of the quantifier. Further, both occurrences of x in $(\forall x)P(x, y)$ are bound occurrences and that of y is a free occurrence.

Consider another example $(\forall x)(P(x) \rightarrow (\exists y)R(x, y))$. The scope of x is $P(x) \rightarrow (\exists y)R(x, y)$. Again here scope of $(\exists y)$ is $R(x, y)$. Here all occurrences of x and y are bound occurrences.

In the formula $(\exists x)P(x) \wedge Q(x)$, the scope of the quantifier $(\exists x)$ is $P(x)$. Here the occurrence of x in $Q(x)$ is free.

Example

Write the predicate expression for 'x is the father of the mother of y.'
Ans: Let us denote

$$P(x) : x \text{ is a person}$$
$$F(x, y) : x \text{ is the father of } y$$
$$M(x, y) : x \text{ is the mother of } y$$

Here x is the father of the person who is the mother of y. Let z denotes this person. Then x is the father of z and z is the mother of y. Hence from the given data, we deduce that there is a person z such that x is the father of z and z is the mother of y. In notations

$$(\exists z)(P(z) \wedge P(x, z) \wedge M(z, y)).$$

Example

Write the predicate expression for 'If x is taller than y, then y is not taller than x.
Ans: Here let $P(x)$ denotes the statement function 'x is a person' and $T(x, y)$ denotes the statement function 'x is taller than y. In symbols, the given expression can be

$$(\forall x)(\forall y)((P(x) \wedge P(y)) \wedge (T(x, y) \rightarrow \neg T(y, x))).$$

Example

Write the predicate for the expression 'All the world loves a lover'.

Ans: Here the statement means that all people love all those people who are lovers. Thus if x is any person and y is any another person who is also a lover, then x loves y. Let

$$\begin{aligned} P(x) &: x \text{ is a person} \\ L(x) &: x \text{ is lover} \\ R(x,y) &: x \text{ loves } y \end{aligned}$$

Using these notations, the given statement can be expressed as

$$(\forall x)(P(x) \to (y)(P(y) \wedge L(y) \to R(x,y))).$$

The Universe of Discourse

If the variables that are quantified in a predicate formula stand for only those objects that are members of a particular class, then the restricted class is called the *universe of discourse*.

For example, consider the statement 'All men are giants'.
Here if $G(x)$: x is a giant and $M(x)$: x is a man, then the given statement can be $(\forall x)(M(x) \wedge G(x))$.

Here if the universe of discourse is considered to be the set of men, then the statement can be $(\forall x)G(x)$.

Example

'Given any positive integer, there is a greater positive integer',symbolize this statement with and without using the set of positive integers as the universe of discourse.
Ans: Let x and y be restricted to the set of positive integers. Then if $G(x,y)$ means x is greater than y, the statement can be $(\forall x)(\exists y)G(y,x)$.

If the universe is not restricted, then if $P(x)$ means x is a positive integer, then the statement can be $(\forall x)(P(x) \to (\exists y)(P(y) \wedge G(y,x)))$.

Exercises

1. Indicate the variables that are free and bound in the statement function '$(\forall x)(P(x) \wedge R(x)) \to (\forall x)(P(x) \wedge Q(x))'$. Also show the scope of the quantifiers.

2. Find the truth values of $(\forall x)(P(x) \vee Q(x))$ where $P(x) : x = 1, Q(x) : x = 2$ and the universe of discourse is $\{1, 2\}$.