DISCRETE STRUCTURES

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Chapter 1

Set theory

Introduction

Objects can be classified according to their common characteristics. These classifications give rise to the concept of sets in mathematics.

Set

Set is a collection of well defined distinct objects. Well defined means that it is possible to decide if a given object belongs to the collection or not. Also, distinct means that members of a set are all different. Members of a set are called *elements*. Generally, capital letters A, B, C, etc are used to denote sets and small letters p, q, r, etc are used to denote elements in a set.
If p is an element in a set A, then it is denoted by

$$p \in A.$$

If q is not an element in a set B, then it is denoted by

$$q \notin B.$$

Members of a set can be sets also. In this case the set under consideration will be set of sets. Similarly set of sets of sets can also be defined and this can be extended further.

Representation of sets

A set can be represented in different ways. One way of describing a set is by listing the elements of the set between braces. For example, the set consisting of numbers $1, 2, 3, 4$ can be written as $\{1, 2, 3, 4\}$. If this set is denoted by A, then we write

$$A = \{1, 2, 3, 4\}.$$

Note that elements in a set are separated by comma. Also the elements of a set can be listed in any order.

Another method of representation of sets is using predicates. A predicate is a property that holds for all the elements in a particular collection. If P denotes a property, then $\{x|\ P(x)\}$ denotes the set of all elements x that satisfy the property P. For example, if P denotes the property that the elements are odd numbers between zero and ten, then the set

$$\{x|\ P(x)\} = \{1, 3, 5, 7, 9\}.$$

The set $A = \{1, 2, a\}$ can be denoted using the predicate in the form

$$A = \{x|\ (x = 1) \vee (x = 2) \vee (x = a)\}.$$

Here notice that the symbol $\vee$ is used to denote the word 'or'. Also the symbol $\wedge$ is used to denote the word *and* in set theory.

Set inclusion

If A and B are two sets such that every element of A is an element in B then A is called a *subset* of B. This is denoted by $A \subseteq B$. If A and B are two sets such that $A \subseteq B$ and $B \subseteq A$, then the sets A and B are said to be equal. This is denoted by $A = B$.

Notice that a set is a subset of itself. If A and B are two sets such that $A \subseteq B$ and $A \neq B$, then A is called a proper subset of B. If A is a proper subset of B, it is denoted by $A \subset B$.

For example, if $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5, a\}$, then it follows that $A \subset B$.

Universal set, Empty set

A set which contains every set under consideration is called a *universal set.* Universal set is generally denoted by U. From the definition, it follows that every set is a subset of a universal set.

A set which does not contain any element is called an *empty set.* Empty set is also called *null set* and is usually denoted by $\emptyset$.

For example, the set of all even prime numbers that are greater than two is an empty set. Empty set $\emptyset$ is a subset of every set.

Power set

Given any set A, the *power set* of A denoted by $\rho(A)$ is the set of all subsets of A. Power set of a set A is also denoted by 2^A. Thus

$$\rho(A) = 2^A = \{B|\ B \subseteq A\}.$$

For example, if $A = \{a, b\}$, then $\emptyset, S_1 = \{a\}, S_2 = \{b\}, S_3 = \{a, b\}$ are all subsets of A. Thus $\rho(A) = \{\emptyset, S_1, S_2, S_3\}$. Here A contains two elements and $\rho(A)$ contains $4 = 2^2$ elements. Generally we have

Theorem 1. *If a set A contains n elements, then power set $\rho(A)$ contains 2^n elements.*

Proof. *Here the theorem is proved using induction method. We have seen the result that a set containing 2 elements have 2^2 subsets. This shows that the theorem is true when $n = 2$.*

Now assume that the result is true for all sets where number of elements are equal to m. Let A contains $m + 1$ elements. Remove one element from A and consider the set of all subsets of the containing the remaining m elements. By assumption, number of subsets of this set is 2^m. If we attach the removed element to each of these 2^m sets, then the new sets will be different from each of the sets already considered. Now altogether there will be $2^m + 2^m = 2 \times 2^m = 2^{m+1}$. subsets. This proves that the result is true when $n = m+1$. Hence by induction, the theorem is proved. □

Example

Find the power set of the set $A = \{1, 2, 3\}$.
Ans: Here subsets of the set A are $\emptyset, S_1 = \{1\}, S_2 = \{2\}, S_3 = \{3\}, S_4 = \{1, 2\}, S_5 = \{2, 3\}, S_6 = \{1, 3\}, S_7 = \{1, 2, 3\}$. Hence the power set contains 8 elements and power set $\rho(A) = \{\emptyset, S_1, S_2, S_3, S_4, S_5, S_6, S_7\}$.

Operation on sets

As in the case of numbers, sets also can be combined using various set operations.

If A and B are two sets, then *union* of these sets denoted by

$$A \cup B = \{x |\ (x \in A) \vee (x \in B)\}.$$

For example, if $A = \{1, 2, 3\}$ and $B = \{2, 3, 4, a\}$ then $A \cup B = \{1, 2, 3, 4, a\}$. From the definition, it follows that union of two sets contain those elements that are in the sets put together common elements, if any, considered only once.

If A and B are two sets and $\emptyset$ is the empty set, then $A \cup B = B \cup A, A \cup \emptyset = A$. Further if C is another set, then $A \cup (B \cup C) = (A \cup B) \cup C$. Using this, if $\{A_1, A_2, \cdots, A_n\}$ is a collection of n sets, then their union is $A_1 \cup A_2 \cup \cdots \cup A_n = \cup_{i=1}^n A_i$. Further, $x \in \cup_{i=1}^n A_i$ if $x \in A_i$ for some i such that $1 \leq i \leq n$.

If A and B are two sets, their *intersection* denoted by

$$A \cap B = \{x|(x \in A) \wedge (x \in B)\}.$$

For example, if $A = \{1, 2, 3\}$ and $B = \{2, 3, 4, a\}$ then $A \cap B = \{2, 3\}$. From the definition, it follows that intersection of two sets contain those elements that are common to both sets.

If A and B are two sets such that $A \cap B = \emptyset$, then the sets A and B are called *disjoint*.

If A and B are two sets and $\emptyset$ is the empty set, then $A \cap B = B \cap A, A \cap \emptyset = \emptyset$. Further if C is another set, then $A \cap (B \cap C) = (A \cap B) \cap C$. Using this, if

$\{A_1, A_2, \cdots, A_n\}$ is a collection of n sets, then their intersection is $A_1 \cap A_2 \cap \cdots \cap A_n = \cap_{i=1}^{n} A_i$. Further, $x \in \cap_{i=1}^{n} A_i$ if $x \in A_i$ for all i such that $1 \leq i \leq n$.

If A is any set, then *complement* of A denoted by

$$A' = \{x |\ (x \in U) \wedge (x \notin A)\}.$$

From the definition of complement, it follows that $A \cup A' = U$ and $A \cap A' = \emptyset$.

Let A and B be any two sets. Then *difference* of these sets

$$\begin{aligned} A - B &= \{x |\ (x \in A) \wedge (x \notin B)\} \\ B - A &= \{x |\ (x \in B) \wedge (x \notin A)\}. \end{aligned}$$

From definition it follows that $A - B = B - A$ only if $A = B$. Also, $A - B = A \cap B'$ and $B - A = B \cap A'$.

Let A and B be any two sets. Then *symmetric difference* of these sets denoted by

$$A \triangle B = (A - B) \cup (B - A).$$

For example, if $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$, then

$$\begin{aligned} &A - B = \{1, 2\} \text{ and } B - A = \{5, 6\} \\ &A \triangle B = \{1, 2, 5, 6\}. \end{aligned}$$

Example

Show that $(A - C) - (B - C) = (A - B) - C$

Ans:

$$\begin{aligned} (A - C) - (B - C) &= (A \cap C') - (B \cap C') \\ &= (A \cap C') \cap (B \cap C')' \\ &= (A \cap C') \cap (B' \cup C) \\ &= ((A \cap C') \cap B') \cup ((A \cap C') \cap C) \\ &= ((A \cap (C' \cap B')) \cup (A \cap C') \cap C)) \\ &= (A \cap (B' \cap C')) \cup (A \cap \emptyset) \\ &= ((A \cap B') \cap C') \cup \emptyset \\ &= (A - B) \cap C' \\ &= (A - B) - C. \end{aligned}$$

Example

Show that if $A \cup B = A \cup C,\ A \cap B = A \cap C$ then $B = C$.

Ans:

$$\begin{aligned} B &= B \cup (A \cap B) \\ &= B \cup (A \cap C) \\ &= (B \cup A) \cap (B \cup C) \\ &= (A \cup B) \cap (B \cup C) \\ &= (A \cup C) \cap (B \cup C) \\ &= (A \cap B) \cup C \\ &= (A \cap C) \cup C \\ &= C. \end{aligned}$$

Exercises

1. Let $U = \{a, b, c, d, e, f, g, k\}, A = \{a, b, c, g\}, B = \{d, e, f, g\}$ and $C = \{a, c, f\}$. Compute $A \cup B$, $A \cap C$, $(A \cup B) - C$, A' and $A \Delta B$.
2. Let $A = \{a, b, c, g\}, B = \{d, e, f, g\}$ and $C = \{a, c, f\}$. Verify that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
3. Let U be the set of rel numbers. $A = \{x |\ x \text{ is a solution of } x^2 - 1 = 0\}$ and $B = \{-1, 4\}$. Compute $(A \cup B)'$ and $(A \cap B)'$.
4. If $A_1 = \{1, 2\},\ A_2 = \{2, 3\},\ A_3 = \{1, 2, 3, 6\}$, what are $\cup_{i=1}^{3} A_i$ and $\cap_{i=1}^{3} A_i$?
5. Show that $(A \cap B) \cup C = A \cap (B \cup C)$ if and only if $C \subseteq A$.
6. Show that $A \subseteq B$ if and only if $A \cap B = A$.
7. Prove that $A - B = A \cap B' = B' - A'$.
8. Show that $A \subseteq B$ if and only if $B' \subseteq A'$.
9. Show that for any two sets A and B, $A - (A \cap B) = A - B$.
10. Prove that for any three sets A, B and C, $(A \Delta B) \Delta C = A \Delta (B \Delta C)$.
11. Prove that $(A - B) - C = A - (B \cup C)$
12. Prove that $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$
13. Prove that $A \Delta B = (A \cup B) - (A \cap B)$
14. $(A \cup C) \subseteq (B \cup C)$ and $(A \cup C') \subseteq (B \cup C')$, show that $A \subseteq B$.

Cartesian products

Let A and B be two sets. The *Cartesian product* of A and B denoted by

$$A \times B = \{(x, y) | (x \in A) \wedge (y \in B)\}.$$

The element (x, y) is called an ordered pair. Note that $(x, y) \neq (y, x)$ and $(x, y) = (y, x)$ only if $x = y$.

Example

Let $A = \{a, b\}$ and $B = \{1, 2, 3\}$. In this case

$$\begin{aligned} A \times B &= \{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)\} \\ B \times A &= \{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\} \\ A \times A &= \{(a,a),(a,b),(b,a),(b,b)\} \\ B \times B &= \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\} \end{aligned}$$

Notice that $A \times B \neq B \times A$

Example

Let $A = \{a, b\}, B = \{1, 2\}$ and $C = \{2, 3\}$. then $B \cup C = \{1, 2, 3\}$ and $B \cap C = \{2\}$ so that

$$\begin{aligned} A \times (B \cup C) &= \{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)\} \\ A \times B &= \{(a,1),(a,2),(b,1),(b,2)\} \\ A \times C &= \{(a,2),(a,3),(b,2),(b,3)\} \\ (A \times B) \cup (A \times C) &= \{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)\} \\ A \times (B \cap C) &= \{(a,2),(b,2)\} \\ (A \times B) \cap (A \times C) &= \{(a,2),(b,2)\}. \end{aligned}$$

Example

For any three sets A, B and C, prove that

$$\begin{aligned} A \times (B \cup C) &= (A \times B) \cup (A \times C) \text{ and} \\ A \times (B \cap C) &= (A \times B) \cap (A \times C) \end{aligned}$$

Ans: Let $p \in A \times (B \cup C)$. Then $p = (x, y)$ where $(x \in A) \wedge (y \in B \cup C)$. That is $(x \in A) \wedge ((y \in B) \vee (y \in C))$, which implies $(x \in A, y \in B) \vee (x \in A, y \in C)$. This give $((x, y) \in A \times B) \vee ((x, y) \in A \times C)$. Thus $(x, y) \in (A \times B) \cup (A \times C)$. Hence $p \in (A \times B) \cup (A \times C)$ so that $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$. Similarly, we can show that $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$ and hence the result.

Second result can also be proved in a similar manner.

Relations

The word relation generally indicates a family tie between two people such as Rahul is the son of Meena, Meena is the sister of Gopu, Rahul is the grandson of Gopal, etc. The concept of relation can very well be carried over to mathematical objects also.

Let A and B be two sets. Then a *relation* from A to B is a rule by which the elements of A are associated to the elements of B. A relation between two sets can also be defined by listing the associated elements as ordered pairs. Using this, a *relation* from A to B is defined as a subset of $A \times B$. In particular, a relation on a set A is a subset of $A \times A$.

For example, let $A = \{a, b\}$ and $B = \{1, 2, 3\}$. Then $R = \{(a, 1), (a, 2), (b, 1)\}$ is a relation from A to B. Here since $(a, 1) \in R$, a is said to be R related to 1. Similarly, a is R related to 2 and b is R related to 1.

If A has m elements and B has n elements, then $A \times B$ has mn elements. By definition, a relation from A to B is a subset of $A \times B$. Also the number of subsets of $A \times B$ is the number of elements in $\rho(A \times B)$. But $\rho(A \times B)$ contains 2^{mn} elements. Hence if A has m elements and B has n elements, then the number of relations from A to B is equal to 2^{mn}.

Domain and Range

If R is a relation from A to B, then A is called *domain* of R and is denoted by $D(R)$. The set B is called *range* of R and is denoted by $R(R)$.

If the relation R is defined on a set A, then its domain and range are the same and is the set A.

Equivalence relations

Let A be a set and R be relation on A. Then R is said to be *reflexive* if $(a, a) \in R$ for all $a \in A$.

For example, if the set $A = \{1, 2\}$, then $R = \{(1, 1), (2, 2)\}$ is a reflexive relation on A. In the set of real numbers, $\leq$ is a reflexive relation.

A relation R on A is said to be *irreflexive* if for every $a \in A, (a, a) \notin R$.

For example, in the set $A = \{1, 2\}$, the relation $R = \{(1, 2), (2, 1)\}$ is irreflexive relation on A. In the same set $R = \{(1, 1), (1, 2), (2, 1)\}$ is not reflexive and not irreflexive.

A relation R on a set A is *symmetric* if whenever $(a, b) \in R$ then (b, a) is also in R.

For example, $R = \{(1, 2), (2, 1)\}$ is a symmetric relation on the set $A = \{1, 2\}$. The relation of being brother is a symmetric relation in the set of all males.

A relation R on a set A is *antisymmetric* if $(a, b) \in R$ and $(b, a) \in R$ implies $a = b$.

In the set of real numbers, $\leq$ is antisymmetric relation.

A relation R on a set A is *transitive* if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

If $A = \{1, 2\}$, the relation $R = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ is a transitive relation on A.

A relation which is reflexive, symmetric and transitive is called an *equivalence relation*

In $A = \{1, 2\}$, the relation $R = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ is an example of an equivalence relation.

Example

Show that similarity of triangles in a plane is an equivalence relation.
Ans: Let S denotes the set of all triangles in a plane. p and q be two triangles in S and R be the relation defined by the rule pRq if 'p is similar to q'.

A triangle is similar to itself. So pRp for any p in S. Hence R is reflexive. Again, let p and q be two triangles such that pRq. Thus p is similar to q. Then q is similar to p so that qRp. Hence R is symmetric. Further, if p, q and r are three triangles such that pRq and qRr, then by definition p is similar to q and q is similar to r. Then p is similar to r so that pRr. So R is transitive and hence the relation R is an equivalence relation.

Example

Let $\mathcal{Z}$ be the set of integers and R be the relation called 'congruence modulo m' defined by

$$R = \{(x, y)|x \in \mathcal{Z} \wedge y \in \mathcal{Z} \wedge (x - y) \text{ is divisible by } m\}$$

Show that R is an equivalence relation.
Ans: For $x \in \mathcal{Z}$, $x - x = 0$ is divisible by m. Hence $(x, x) \in R$ for all $x \in \mathcal{Z}$. Thus R is reflexive. Again, let x and y be two integers such that $(x, y) \in R$. Thus $x - y$ is divisible by m. Now, $(y - x) = -(x - y)$ is also divisible by m. Hence R is symmetric. Further, if x, y and z are three integers such that $(x, y) \in R$ and $(y, z) \in R$, then by definition $x - y$ and $y - z$ are divisible by m. Hence $x - y + y - z = x - z$ is divisible by m, which gives $(x, z) \in R$. So R is transitive and hence the relation R is an equivalence relation.

Example

The following rules also define equivalence relations:

(1) Equality on the set of real numbers

(2) Equality of subsets of a universal set

(3) Relation of lines being parallel on the set of lines in a plane

(4) Relation 'living in the same town'on the set of people in a state.

Compatibility

A relation which is reflexive and symmetric called a *compatibility relation*. From definition it follows that all equivalence relations are compatibility relations. But the converse is not true. For example in the set $A = \{\text{ball, bed, dog, let, egg}\}$, the relation

$$R = \{(x, y)|\ (x, y \in X) \wedge (xRy \text{ if } x \text{ and } y \text{ contain a common letter})\}$$

is an example of a compatibility relation that is not an equivalence relation.

Composition of relations

If Rahul is the son of Meena and Meena is the sister of Gopu, then Rahul is the nephew of Gopu. Again, if Meena is the daughter Gopal, then Rahul is the grandson of Gopal. This means that two or more relations can be combined to obtain another relation. This method of composition of relations can be carried over to mathematical relations also.

Let R be a relation from X to Y and S be a relation from Y to Z. The a relation $R \circ S$ is a relation from X to Z defined by

$$R \circ S = \{(x, y) | x \in X \wedge z \in Z \wedge (\exists y)(y \in Y \wedge (x, y) \in R \wedge (y, z) \in S\}.$$

This relation $R \circ S$ is called *composition* of the relations R and S.

Example

Let $R = \{(1, 2), (3, 4), (2, 2)\}$ and $S = \{(4, 2), (2, 5), (3, 1), (1, 3)\}$. Find $R \circ S, S \circ R, R \circ (S \circ R), (R \circ S) \circ R, R \circ R, S \circ S$, and $R \circ R \circ R$.
Ans:

$$\begin{aligned} R \circ S &= \{(1, 5), (3, 2), (2, 5)\} \\ S \circ R &= \{(4, 2), (3, 2), (1, 4)\} \\ R \circ (S \circ R) &= \{(3, 2)\} \\ (R \circ S) \circ R &= \{(3, 2)\} \\ R \circ R &= \{(1, 2), (2, 2)\} \\ S \circ S &= \{(4, 5), (3, 3), (1, 1)\} \\ R \circ R \circ R &= \{(1, 2), (2, 2)\} \end{aligned}$$

Example

R and S be two relations on the set of positive integers $\mathcal{I}$ such that $R = \{(x, 2x) | \ x \in \mathcal{I}\}$ and $S = \{(x, 7x) | \ x \in \mathcal{I}\}$. Find $R \circ S, R \circ R, R \circ R \circ R, R \circ S \circ R$.
Ans:

$$\begin{aligned} R \circ S &= \{(x, 14x) | \ x \in \mathcal{I}\} \\ R \circ R &= \{(x, 4x) | \ x \in \mathcal{I}\} \\ R \circ R \circ R &= \{(x, 8x) | \ x \in \mathcal{I}\} \\ R \circ S \circ R &= \{(x, 28x) | \ x \in \mathcal{I}\} \end{aligned}$$

Partition and covering of a set

Let S be a given set and $A = \{A_1, A_2, \cdots, A_n\}$ where each $A_i, i = 1, 2 \cdots, n$ is a subset of S and $\cup_{i=1}^{n} A_i = S$. Then A is called a *covering* of S. Moreover if $A_i, i = 1, 2 \cdots, n$ are mutually disjoint, then A is called a partition of S.

For example if $S = \{1, 2, 3\}$, then $A_1 = \{1\}, A_2 = \{2\}, A_1 = \{3\}$ is a partition of S. $B_1 = \{1\}, B_2 = \{2, 3\}$ is another partition of S.

Exercises

1. Show by means of an example that $A \times B \neq B \times A$.

2. If $U = \{a, b, c, d, e, f, g\}, A = \{a, d, e, f\}, B = \{b, e, g\}$, and $C = \{a.c.e.g\}$ verify that $A \times (B - C) = (A \times B) - A \times C)$

3. Let $A = \{1, 2\}$. Construct the set $\rho(A) \times A$.

4. In the set of all people in a country, define a relation R by the rule aRb if a and b speak the the same language. Prove that R is an equivalence relation.

5. Give an example of a relation that is neither reflexive nor symmetric.

6. Give an example of a relation which is reflexive, symmetric but not transitive.

7. Given $S = \{1, 2, \cdots, 10\}$ and relation R on S where $R = \{(x, y)|\ x+y = 10\}$. What are the properties of R ?

8. Let R be a relation on the set of all positive integers such that $R = \{(x, y)|\ x-y$ is an odd positive integer $\}$. Verify whether R is an equivalence relation.

9. On $X = \{a, b, c\}$, let $R_1 = \{(a, b), (a, c), (c, b)\}$ and $R_2 = \{(a, b), (b, c), (c, a)\}$. Then find $R_1 \circ R_1, R_1 \circ R_2, R_2 \circ R_1, R_2 \circ R_2$.

10. If R_1 and R_2 are two equivalence relations on A, then prove that $R_1 \cup R_2$ is also an equivalence relation on A.

11. Give an example of a covering of the set $A = \{1, 2, 3, 4\}$.

12. Find all partitions of the set $A = \{a, b, c\}$.

Functions

Functions are all relations but all relations need not be functions.

Let X and Y be two sets. A relation from X to Y is called a *function* if for every $x \in X$, there exist a unique $y \in Y$ such that $(x, y) \in f$. Functions are also called mappings or correspondence.

If f is a function from X to Y, then it is denoted as $f : X \to Y$. Moreover, $(x, y) \in f$ is denoted like $y = f(x)$. If $y = f(x)$, then y is called the image of x.

$f : X \to Y$ is called *one-to-one* function if $x_1, x_2 \in X, x_1 \neq x_2$ and $(x_1, y_1), (x_2, y_2) \in f$ implies $y_1 \neq y_2$. Equivalently $f : X \to Y$ is one-to-one if $f(x_1) = f(x_2)$, implies $x_1 = x_2$.

One-to-one functions are also called *injections*.

$f : X \to Y$ is called *onto* if for all $y \in Y$ there exist $x \in X$ such that $(x, y) \in f$. Onto functions are also called surjections.

A function which is both one-to-one and onto is called a *bijection*.

Identity function

Let A be an arbitrary set. Define a function i on A by the rule $i(x) = x$. It can be verified that i is a one-to-one function from A onto itself. The function defined thus is called the *identity function* on the set A. Identity function on a set is a bijection.

Example

Let $\mathcal{R}$ denotes the set of real numbers.$f : \mathcal{R} \to \mathcal{R}$ be defined by the rule $f(x) = x + 2$. Show that this is a a function which is a bijection.
Ans: Let $x_1, x_2 \in \mathcal{R}$ be such that $f(x_1) = f(x_2)$. That is $x_1 + 2 = x_2 + 2$. Then clearly, $x_1 = x_2$. Hence by definition f is one-to-one.

Let $x \in \mathcal{R}$, then consider the real number $x-2$. Now $f(x-2) = (x-2)+2 = x$. Hence f is onto. By definition, f is a bijection.

Composition of functions

Let $f : X \to Y$ and $g : Y \to Z$ be two functions. Then composition of these functions denoted by $g \circ f$ is defined by

$$g \circ f = \{(x, z) |\ (x \in X) \wedge (z \in Z)(\exists y)(y \in Y \wedge y = f(x) \wedge z = g(y))\}.$$

If f and g are defined in terms of rules, then$g \circ f$ is defined as $(g \circ f)x = f(g(x))$.

If f is a function on a set A, then $f \circ f$ is also a function on A. This function is denoted as f^2. Similarly we can define f^3, f^4 etc.

Example

$f : \mathcal{R} \to \mathcal{R}$ be defined by $f(x) = x^2 + 1$ and $g : \mathcal{R} \to \mathcal{R}$ be defined by $g(x) = 2x$. Find $f \circ g$ and $g \circ f$.
Ans:

$$(f \circ g)x = g(f(x)) = g(x^2 + 1) = 2(x^2 + 1)$$
$$(g \circ f)x = f(g(x)) = f(2x) = (2x)^2 + 12 = 4x^2 + 1$$

Example

$f : \mathcal{R} \to \mathcal{R}$ be defined by $f(x) = x^2 - 2$ and $g : \mathcal{R} \to \mathcal{R}$ be defined by $g(x) = x + 4$. Find $f \circ g$ and $g \circ f$.
Ans:

$$(f \circ g)x = g(f(x)) = g(x^2 - 2) = x^2 - 2 + 4 = x^2 + 2$$
$$(g \circ f)x = f(g(x)) = f(x + 4) = (x + 4)^2 - 2 = x^2 + 8x + 14$$

Example

Let $X = \{1, 2, 3, 4\}$. Let $f : X \to X$ be given by $f = \{(1, 2), (2, 3), (3, 4), (4, 1)\}$. Find f^2 and f^3.
Ans:

$$f^2 = f \circ f = \{(1, 3), (2, 4), (3, 1), (4, 2)\}$$
$$f^3 = f^2 \circ f = \{(1, 4), (2, 1), (3, 2), (4, 3)\}.$$

Exercises

1. Do the following sets define functions ? If so give their domain and range in each case,

   (a) $\{(1, (2, 3)), (2, (3, 4)), (3, (1, 4)), (4, (1, 4)\}$

   (b) $\{(1, (2, 3)), (2, (3, 4)), (3, (3, 2))\}$

   (c) $\{(1, (2, 3)), (2, (3, 4)), (1, (2, 4))\}$

   (d) $\{(1, (2, 3)), (2, (2, 3)), (3, (2, 3))\}$

2. List all possible functions from $X = \{a, b, c\}$ to $Y = \{0, 1\}$ and indicate in each case whether the function is one-to-one onto.

3. Let $X = \{1, 2, 3\}, Y = \{p, q\}$ and $Z = \{a, b\}$. Also let $f : X \to Y$ be $f = \{(1, p), (2, p), (3, q)\}$ and $g : y \to Z$ be given by $g = \{(p, b), (q, b)\}$. Find $g \circ f$.

4. Show that there exists a one to one function from $A \times B$ to $B \times A$. Is this function onto ?

5. If $f(x) = \sin x$ and $g(x) = 2x + 1$, find $f \circ g$ and $g \circ f$.

6. If $f : A \to B$ and $g : B \to C$ are two functions, show that $g \circ f$ is onto if g is onto.

Cardinality of a set

A set X is said to be *finite* if there is an integer n such that there is a bijection from X onto $\{1, 2, \cdots, n\}$. If a set is not finite, it is said to be an *infinite* set.

$\{1, 2, 3, 4, 5\}$ is an example of a finite set. $\{x|\ x$ is a prime number$\}$ is an example for an infinite set. *Cardinality* of a finite set is the number of elements in the set. If a set is not finite, cardinality of the set is infinity. Cardinality of a set A is denoted $|A|$.

Principle of inclusion and exclusion

The following theorem gives aa equation for the cardinality of union of two finite sets.

Theorem 2. *Let A and B be any two finite sets. Then $|A \cup B| = |A| + |B| - |A \cap B|$.*

Proof. *Let the set A contains m elements and B contains n elements. Also assume that the sets contain no common elements. Then $A \cup B$ contains $m+n$ elements and all these elements are distinct. Hence $|A \cup B| = m+n = |A|+|B|$. Here $A \cap B = \emptyset$ so that $|A \cap B| = 0$.*

On the other hand, if A and B contains p elements in common, then in $A \cup B$, these p elements will be duplicated. If an element is duplicated, it will be counted only once in $A \cup B$. Hence the number of elements in $A \cup B$ will be equal to the the number of elements in A plus the number of elements in B minus the number of elements that are duplicated. Number of elements that are duplicated is the number of elements that are in $A \cap B$. Thus $|A \cup B| = |A|+|B|-|A \cap B|$. □

Corollary 1. *If A, B and C are any three sets, then $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$.*

In the general case, if $A_1, A_2, \cdots, A_n$ are n sets, then

$$\begin{aligned}|A_1 \cup A_2 \cup \cdots \cup A_n| = &\sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| \\ &+ \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| + \cdots \\ &+ (-1)^n |A_1 \cap A_2 \cdots \cap A_n|.\end{aligned}$$

This rule is called the principle of inclusion and exclusion.

Example

Out of 200 students, 50 of them take the course Discrete Mathematics, 140 of them take the course Economics, and 24 of them take both courses. Both courses have scheduled examinations for a particular day. How many students will be free on that day ?

Ans: Let A denote the set of students who take the course Discrete Mathematics and B denote those take the course Economics. Then $|A| = 50, |B| = 140$ and $|A \cap B| = 24$. Students who have examination on that particular day are those included in $A \cup B$. Now $|A \cup B| = |A|+|B|-|A \cap B| = 50+140-24 = 166$. Number of students who are free on that day is $200 - 166 = 34$.

Countably infinite set

An infinite set A is called *countably infinite* if there is a bijection from the set of natural numbers onto A.

Since the identity map is a bijection on every set, it follows that the set of natural numbers itself is a countably infinite set.

Since the function $f(x) = 2x$ is a bijection from the set of natural numbers to the set of even integers, the set of all even integers is countably infinite. In general, the set of all multiples of an integer m is countably infinite, because the function $f(x) = mx$ is a bijection.

Uncountably infinite set

An infinite set that is not countably infinite is called an *uncountable set*.

Theorem 3. *The set of real numbers between* 0 *and* 1 *is an uncountable set.*

Proof. *If possible, let the set be countably infinite. So there is a bijection from the the set of natural numbers to this set. Hence we can exhaustively list them one after another in decimal form as in the following:*

$$0.a_{11}a_{12}a_{13}a_{14}\cdots$$
$$0.a_{21}a_{22}a_{23}a_{24}\cdots$$
$$0.a_{31}a_{32}a_{33}a_{34}\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$0.a_{i1}a_{i2}a_{i3}a_{i4}\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

where $0 \leq a_{ij} \leq 9$ *denotes the* j*th number in the list. Consider the number*

$$0.b_1b_2b_3b_4\cdots$$

where for all i

$$b_i = \begin{cases} 1 & \textit{if} \quad a_{ii} = 9 \\ 9 - a_{ii} & \textit{if} \quad a_{ii} = 0, 1, 2, \cdots 8 \end{cases}$$

From the definition it follows that the number $0.b_1b_2b_3b_4\cdots$ *is a number between* 0 *and* 1. *This number differs from the first number in the first digit, second number in the second digit etc. Hence we conclude that the above list does not exhaust all the real numbers between* 0 *and* 1. *This contradicts the assumption. Hence the theorem is proved.* □

Exercises

1. Show that the following data is wrong. Out of 900 students, it was reported that 700 drive cars, 400 ride bicycle and 150 drive both cars and bicycles.

2. Thirty cars were assembled in a factory.The options available were a radio, an air conditioner, and white-wall tyres. It is known that 15 of the cars have radios, 8 of them have air conditioners, and 6 of them have white-wall tyres. Moreover, 3 of them have all three options. Find out at least how many cars do not have any options at all ?

3. Among 100 students, 32 study mathematics, 20 study physics, 45 study biology. 15 study mathematics and biology, 7 study mathematics and physics, 10 study physics and biology, and 30 do not study any of these subjects.

   (a) Find the number of students studying all the three subjects

   (b) Find the number of students studying exactly one of the three subjects.

4. Show that the set of ordered pairs $\mathcal{N} \times \mathcal{N}$, where $\mathcal{N}$ denotes the set of natural numbers is countable.

5. Prove that the set of real numbers between 1 and 2 is uncountable.

Pigeonhole principle

Pigeonhole principle, also known as shoe box argument or Dirichlet drawer principle, is one of the basic combinatorial results.

Theorem 4. *(Pigeonhole principle) If there are m pigeons and n pigeonholes and if $m > n$, then there must be some pigeonhole occupied by two or more pigeons.*

Proof. *Let $P_1, P_2, \cdots P_m$ be the pigeons and $H_1, H_2, \cdots H_n$ be the pigeonholes. If possible, let all the pigeons be accommodated and no pigeonhole is occupied by two or more pigeons. Let the pigeon P_1 occupies the hole H_1, P_2 occupies the hole H_2, etc, P_n occupies the hole H_n. This can be assumed because, if P_1 occupies the hole say $H_p, 1 \leq p \leq m$, then rename the hole H_p as H_1 and similarly the others also can be renamed. Now the holes have been exhausted and there is no hole to accommodate the $n+1th$ pigeon. This contradicts the assumption that each pigeon has occupied a hole and no whole accommodates more than one pigeon. Hence our assumption is wrong. This means that either all pigeons are not accommodated or at least one hole will be occupied by more than one pigeons. Hence if we assume that all pigeons are accommodated, then at least one hole will be occupied by more than one pigeons.* □

Example

Prove that among eight persons, there are at least two of them born in the same day of the week.

Ans: Let the 8 persons be pigeons and 7 days be the pigeon holes. Then by pigeonhole principle, there will be at least one hole occupied by more than one pigeon. Here this means that there will be at least one day that is occupied by more than one person. So,there are at least two persons born in the same day of the week.

Example

Prove that among 13 persons, there are at least two of them born in the same month.

Ans: Let the 13 persons be pigeons and 12 months be the pigeon holes. Then by pigeonhole principle, there will be at least one hole occupied by more than one pigeon. Here this means that there will be at least one month that is occupied by more than one person. Hence, there are at least two of them born in the same month.

Example

Prove that among six persons, either there are three persons who are mutual friends or there are three persons who are complete strangers to each other.

Ans: Let A be a person in the group. By the pigeonhole principle, there will be three (or more) friends or three (or more) strangers to A. Let B, C, D be the friends of$A$. If any two of B, C, D know each other, then these persons together with A will constitute three friends. On the other hand no two of B, C, D know each other, these three persons are complete strangers. In a similar way we can argue the case when B, C, D are strangers of A.

Exercises

1. Show that among $n + 1$ arbitrarily chosen integers, there are two whose difference is divisible by n.

2. Show that a sequence of $n^2 + 1$ distinct integers, there is either an increasing sub sequence of length $n + 1$ or a decreasing sub sequence of length $n + 1$.

3. Show that if any 11 numbers are chosen from the set $S = \{1, 2, 3, \cdots, 20\}$, then one of them will be a multiple of another.

4. If we select any 14 integers from 1 to 25, then prove that one of them is a multiple of another.